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ARTIFICIAL INTELLIGENCE β-UNZIPPING METHOD IN STRUCTURAL SYSTEM RELIABILITY ANALYSIS

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Abstract—By comparing the β-unzipping method in a structural system reliability analysis with the problem-solving method used in the artificial intelligence (A.I.) technique, the traditional β-unzipping method is updated and used in a system reliability analysis for space frame structures. In this paper, the idea of heuristic state–space search approach of the problem-solving methods in A.I. is employed. The critical failure probability is established and defined. It is taken as the path value, i.e. the heuristic information in problem-solving to identify the critical failure paths automatically by computer. Typical ship structures are analysed to verify the efficiency and accuracy of the proposed method. Copyright © 1996 Elsevier Science Ltd.

NOTATION

a	cross-section area of structure element
E	Young's modulus
F_i	failure events described in eqn (2)
$\{F\}^e$	structural element node force vector
$-\{F'\}^e$	fictitious structural element node force vector
G	shear modulus, $=E/(2+2\mu)$
i_x, i_y, i_z	inertia moment of cross-section of structural element about local x-, y- and z-axis, respectively
k	number of degrees of redundancy
$[K']^e$	revised structural element stiffness matrix
M_i	failure events described in eqn (4)
n	number of potential failure element in structural system
N_i, N_m	number of failure path at level i and mechanism level
P_i	failure events described in eqn (3)
P_{fs}	system failure probability
$P(.)$	probability of event (.)
${}^kP_{fcr}$	critical failure probability with k degrees of redundancy
R	capacity of failure element
β	reliability index
$\Delta\beta_i$	experiential constant for unzipping at level i
${}^k\beta_{cr}$	critical reliability index with k degrees of redundancy
$\{\delta\}^e$	structural element node displacement vector
μ	Poisson ratio
$\Phi(.)$	standard normal distribution function
$\Phi^{-1}(.)$	reverse function of $\Phi(.)$

1. INTRODUCTION

With the development of the structural reliability theory, many approaches for the structural system reliability analysis have appeared in the literature, e.g. *branch and bound method, β-unzipping method and load incremental method* [1–4], etc. All of these methods place emphasis on modelling the structural system with the combination of different failure paths, and then calculating the system failure probability. The difficulty encountered in these studies lies mainly in the lack of knowledge of identifying the critical failure paths. The critical failure paths are those which give a remarkable contribution to the system failure probability. Lack of this kind of knowledge will cause the combination explosion phenomenon of the failure path which then makes the system reliability analysis impractical.

Many efforts have been made to improve this situation [4–6]. In Ref. [4], the *load incremental method* was updated, but it still cannot deal efficiently with the complicated combination of external loads. Furthermore, the process to obtain the failure paths in this method lacks the probability background. In Refs [5] and [6], the *branch and bound method* was used to solve the system failure probability for plane and space truss structures. The branch and bound process which is used to seek the failure paths is complicated and it is difficult to use the computer program to generate the failure paths automatically. However, considerable work is still required in this area before reliable criteria for the identification of the critical failure paths can be developed.

The process to generate the failure paths in the β-unzipping method is very similar to that of state–space search in the problem-solving methods in artificial intelligence (A.I.) [7]. Thus the concept of the heuristic state–space search method [7, 8] may be employed in the traditional β-unzipping method. The solving graph in the problem-solving method can then be taken as the critical failure paths.

In this paper an attempt is made to combine the β-unzipping method with the technique of problem-solving in A.I. The concept of critical failure

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probability is established and used as the path value, i.e. the heuristic information to identify the critical failure paths in the structural system. By employing this heuristic information, a computer program procedure is developed to generate the system reliability analysis model automatically for the space frame structures. In order to assess efficiency of the proposed method and the accuracy of the program procedure, we have calculated some typical structures. The results are compared with those in the existing literature [1, 9]. Significant agreements are found. Furthermore, the computation effort is much reduced with the help of the method proposed in the present paper.

2. SIMPLE DESCRIPTION OF β-UNZIPPING METHOD

The most typical failure mode of a frame structural is the forming of plastic hinges at one or both ends of the structural elements [1–4]. When a statically undetermined structural system has one such kind of plastic hinge, the stress and strain will be redistributed within the system. However, the structure as a whole still has the capacity to withstand the external loads. As the loads increase, more and more plastic hinges will be formed on its structural elements until the structure turns into a mechanism, when the structural system then collapses and fails to serve its original purpose. To calculate the structural system failure probability at the mechanism level, we should first identify all the critical failure paths which have a relatively large failure probability. After the critical paths have been identified, they are combined into a series system to solve the system reliability [1–4, 10]. The former is crucial. This paper is proposed to develop a criteria to solve the problem.

The most attractive advantage of the β-unzipping method lies in its flexibility to obtain the structural system reliability at any desired level. The entire analysis process begins from level 0; with the operation of unzipping going on, the system reliability at a higher level can be obtained until the mechanism level is reached. The unzipping process can be simply described as follows:

(1) At level 0—at this level, it is similar to the reliability analysis for a single structural element. Supposing that the system has n potential failure elements, which have the reliability indexes of $\beta_i, i = 1, 2, \ldots, n$. The system reliability index is

$$\beta_S = \mathrm{Min}\{\beta_1, \beta_2, \ldots, \beta_n\} = \beta_{\min}. \tag{1}$$

(2) At level 1—given the value of $\Delta\beta_1$, the failure elements whose reliability indexes fall into the interval $[\beta_{\min}, \beta_{\min} + \Delta\beta_1]$ are called the critical failure elements. Let the system have N_1 critical elements, the system failure probability P_{fs} is

$$P_{fs} = P(F_1 \cup F_2, \ldots, \cup F_{N1}), \tag{2}$$

where $P(.)$ is the probability of event (.) and F_i is the failure event of the ith critical failure element.

(3) At level 2—at the condition that the critical failure element i failed, we can calculate the reliability indexes $\beta_{j|i}$ for the rest $(n - 1)$ failure elements. Let the minimum value among all the $\beta_{j|i}$ is $\beta_{j|i,\min}$. By using the constant value $\Delta\beta_2$, we can pick out the failure elements whose $\beta_{j|i}$ falls into the interval $[\beta_{j\,i,\min}, \beta_{j\,i,\min} + \Delta\beta_2]$. Each of them will be combined with the critical failure element i to be a critical pair, which itself is a parallel system with two elements. When the above process is repeated for all the critical failure elements in level 1, we obtain N_2 critical pairs of failure elements. Thus the system failure probability P_{fs} can be written as

$$P_{fs} = P(p_1 \cup p_2, \ldots, \cup p_{N2}), \tag{3}$$

where p_i is the failure extent of the ith critical pair.

With the similar process, we can compute the result at level $N (N > 2)$. The higher the level, the more accurate the system probability will be and in addition, the larger the computational efforts will be made accordingly.

(4) At the mechanism level—according to the above unzipping process, we can produce the analysis model as illustrated in Fig. 1, which contains a number of critical failure mechanisms. If the model contains N_m paths, the system failure probability P_{fs} can be calculated as

$$P_{fs} = P(M_1 \cup M_2, \ldots, \cup M_{Nm}), \tag{4}$$

where M_i is the failure event of the ith path.

From the above description, we can see that the crucial problem of this method is to choose the experiential constant $\Delta\beta_i$ at each level. If the value of the chosen constant is too small, some significant critical failure paths will be lost. Conversely, the larger the value the more failure paths which have a small failure probability will be generated. This will cause the computational efforts to increase tremendously, especially for engineering structures with high redundancy. To solve this problem, the concept of critical failure probability, which is based on the

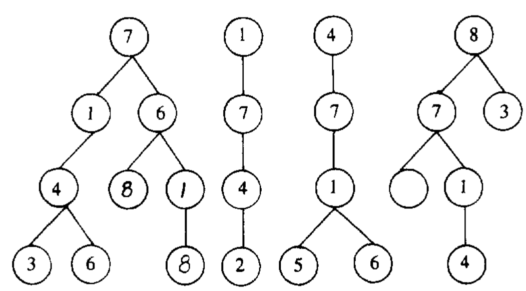

Fig. 1. Tree of critical paths for structural system reliability analysis.

problem-solving method in A.I., is proposed and used as the path value to identify the critical failure paths.

3. PROBLEM-SOLVING METHOD IN A.I. AND THE CRITICAL FAILURE PROBABILITY

As illustrated in Fig. 1, the critical failure paths form a graph which is similar to the solution graph of a state–space search method in A.I. In fact, the whole process of the β-unzipping method is just the breadth-first method of the problem-solving in A.I. [7]. In the problem-solving method, in order to save on computational effort when founding the solution path which has a smaller path value, the most efficient method when using the heuristic information is to employ some criteria to prevent it from generating so many extraneous successive nodes on the searching path at any level. In the β-unzipping method, the experiential value $\Delta\beta_i$ is used for this purpose. On the basis of the idea of problem-solving in A.I. we can establish a specially designated failure probability or its relative reliability index. This particular value has come from the structural system itself and can be used as the heuristic information to identify the critical failure paths.

Let the structural system have k degrees of redundancy, the structural system will then have $(k+1)$ plastic hinges when the global mechanism is formed. According to the structural reliability theory, the failure path is a parallel system which consists of the $(k+1)$ plastic hinges as its elements. The failure probability has the relationship with those of its failure elements and is smaller than that of each single failure element. This value can be taken as the path value to identify the critical failure paths. In the process of the β-unzipping method, the failure elements which has a larger failure probability than the critical value will be branched in the future. Thus the lower bound of this critical value can be used:

$$P_{fsp} = P_{f1} \cdot P_{f2}, \ldots, P_{f(k+1)}. \tag{5}$$

We define the critical failure probability ${}^kP_{fcr}$ as

$$({}^kP_{fcr})^{k+1} = P_{fs}$$

$$({}^kP_{fcr})^{k+1} = \Phi(-\beta_1) \cdot \Phi(-\beta_2), \ldots, \Phi(-\beta_{k+1}), \tag{6}$$

$${}^kP_{fcr} = [\Phi(-\beta_1) \cdot \Phi(-\beta_2), \ldots, \Phi(-\beta_{k+1})]^{1/(k+1)}. \tag{7}$$

The relative critical reliability index ${}^k\beta_{cr}$ is then

$${}^k\beta_{cr} = -\Phi^{-1}({}^kP_{fcr}). \tag{8}$$

In eqns (6)–(8), $\Phi(.)$ is the standard normal distribution function and $\Phi^{-1}(.)$ is its reverse.

The critical reliability index defined by eqn (8) is the equivalent reliability index of a single parallel element in the particular failure path. It can be used in the β-unzipping method at each level. Let the structural system have m degrees of redundancy at level i; we first calculate the critical value ${}^m\beta_{cr}$ according to eqn (8), and consider that those failure elements on its branch which have reliability indexes larger than ${}^m\beta_{cr}$ will not affect the system reliability significantly, i.e. we take $\Delta\beta_i = {}^m\beta_{cr} - \beta_{fi,\min}$. We can thus obtain all the critical failure paths by repeating the above process until the mechanism level is reached. Using this method, we analysed several practical structural systems. Satisfied results are obtained and the computational efforts are reduced remarkably.

4. AUTOMATIC GENERATION OF THE CRITICAL FAILURE PATH

As the continuous emergence of the plastic hinges in the process of identifying the critical failure paths occurs, the entire structure needs to be re-analysed to obtain the updated distribution of its inner forces. Since the practical engineering structures are those with high redundancy, there will exist a lot of plastic hinges in the structures when branching to the mechanism level. In order not to extend the structural stress analysis scale, i.e. not to increase the node and element number, the method that revises the element stiffness matrix is employed [11]. When an element has plastic hinge(s) at its end(s), the element equation can be written as

$$\{F\}^e + (-F'\}^e) = [K']^e \cdot \{\delta\}^e, \tag{9}$$

where $[K']^e$ is the revised stiffness matrix; $(-\{F\}^e)$ is the equivalent fictitious node force vector). When the structure is re-analysed, the stiffness matrix of the elements with the plastic hinges is substituted by the revised one and the fictitious node force is applied to the relative nodes. Some expressions of the revised stiffness matrix and fictitious node forces are listed in the Appendix.

After the global stiffness matrix is assembled, it is checked whether it is singular or not. If it is, a structural mechanism has to be formed, and vice versa. Figure 2 is the flow-diagram of the automatic generation of the critical failure paths.

5. NUMERICAL ANALYSIS AND DISCUSSION

Example 1

As illustrated in Fig. 3, the mean values of the loads are $\mu_{p1} = 55.0$ kN, $\mu_{p2} = 45.0$ kN. The capacity of the elements on the same direction are fully correlated. The mean value of the capacity is $\mu_R = 135$ kN m. The capacity is independent of the loads. All the basic valuables obey the normal

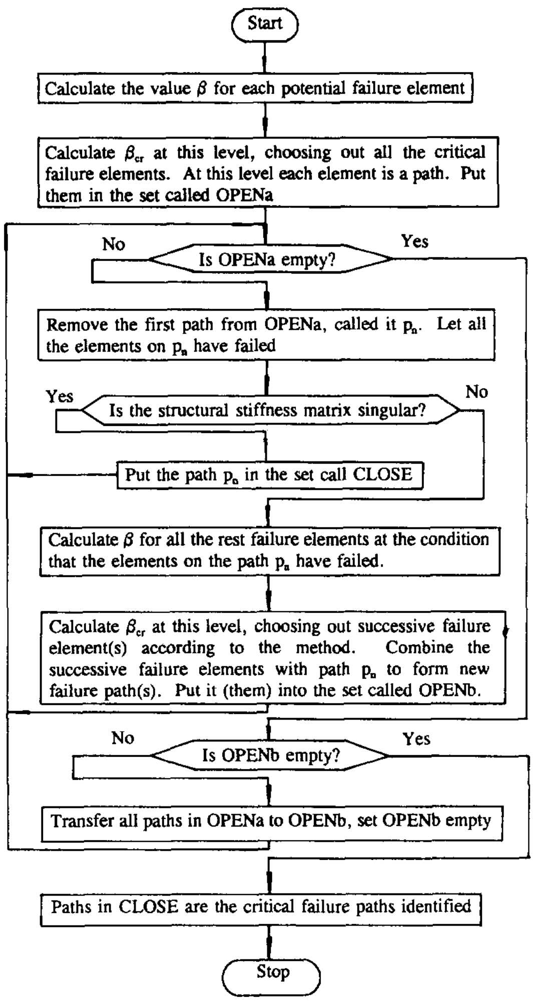

Fig. 2. Flow chart of the program to generate the critical failure paths.

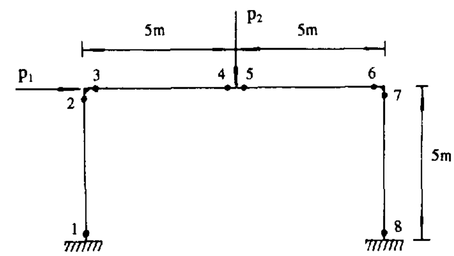

Fig. 3. Reliability analysis model in Example 1; ● potential failure element.

distribution. The coefficient of variance (C.O.V.) is 0.1. The inertia moment of the element is $I = 0.579 \times 10^{-4}\,\mathrm{m}^4$ and Young's modulus is $E = 0.2 \times 10^9\,\mathrm{kN\,m}^{-2}$. The results are listed in Table 1 and compared with those in the literatures. In Table 1, method (1) is that proposed in the present paper. Method (2) is that shown in Ref. [1]. β_d, β_s and β_a are the average reliability indexes obtained from the Ditlevson bound, simple bound and the average correlative parameter method, respectively [1–4]. Figure 4 shows the critical failure paths by using the method proposed in this paper. It can be seen from Table 1 that the number of failure paths is reduced without losing the accuracy.

Example 2

The typical transversal frame of a large oil ship in Ref. [9] is analysed to solve the structural system reliability by using the method proposed in this paper. The ship has the dead weight of 60,000 t. Its wing tanks are fully loaded. The structural model is illustrated in Fig. 5 [9]. The capacities of the structural elements and the external loads are taken as the basic variables and the coefficient of variance of the loads and capacities are $\mathrm{C.O.V.}_L = 0.3$ and $\mathrm{C.O.V.}_R = 0.05$, respectively. All the basic variables obey the standard normal distribution. Table 2 lists all the known parameters.

Figure 6 lists all the critical failure paths identified by the present method. In Fig. 6, the failure path denoted by the arc of "═══" is the dominant one which has the largest failure probability. Those denoted by "---" are the paths which have a relatively smaller failure probability. The failure probability of these kinds of paths are listed in the blanks at the end of the paths, respectively.

The same structure system was calculated in Ref. [9] by employing the branch and bound method to generate the critical failure paths. In the process of generating the safety margin automatically, the correlativity of axial force and transverse shear force in the yield condition was considered. The dominant failure path is listed as ⑨—①—⑩—⑱—⑮—⑤—⑦, which has the failure probability of 0.3494×10^{-2}. From Fig. 6, we can see that the dominant failure path obtained by using the method proposed in this paper is similar to that one except

Table 1. Analysis results for Example 1

Level	Method	$\Delta\beta_i$	Number of paths	β_d β_s β_a
0	(1)	—	—	1.66669
	(2)	—	—	1.67
1	(1)	1.2706	3	1.66 1.62 1.60
	(2)	3.0	5	1.62 1.61 1.60
2	(1)	1.3 1.8 0.4	3	2.51 2.50 2.50
	(2)	—	—	—
Mechanism	(1)	—	3	4.16 4.17 4.17
	(2)	—	—	accurate: 4.19

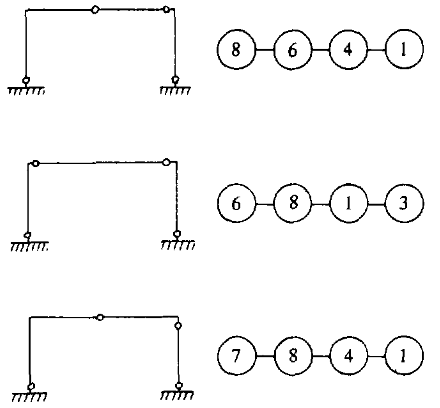

Fig. 4. Critical failure paths and relative collapse model for structure in Example 1.

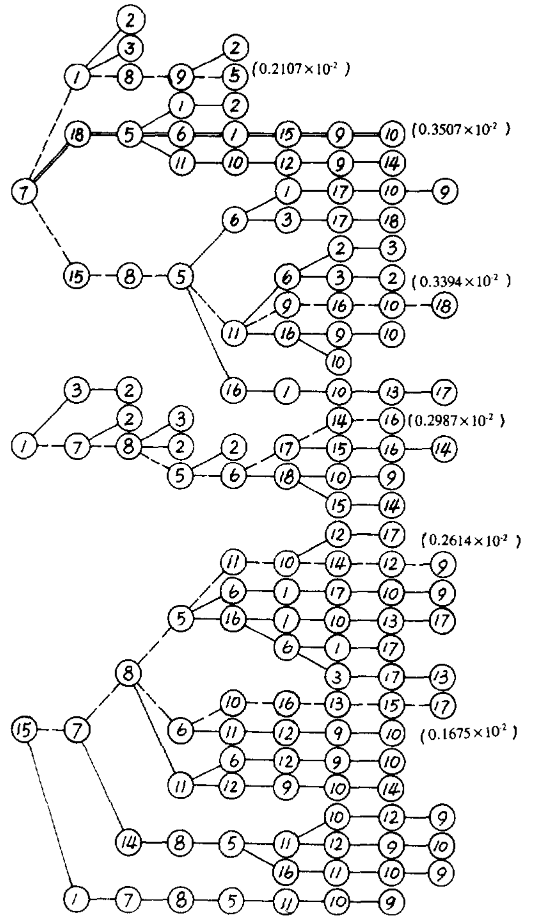

Fig. 6. Critical failure paths in Example 2: ═══ dominant path whose path failure probability is the largest; ---- failure paths with smaller failure probability.

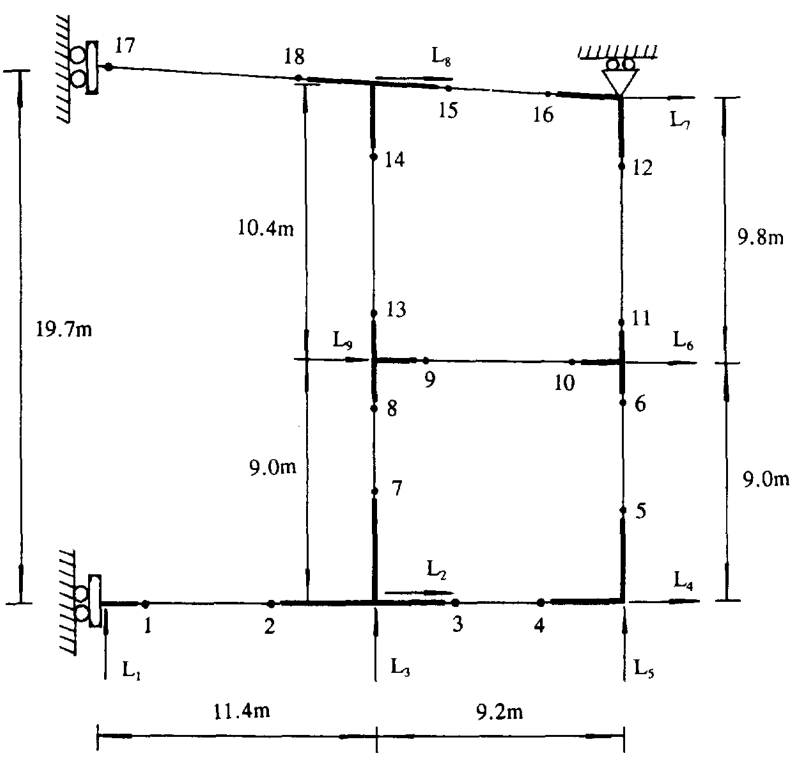

Fig. 5. Reliability analysis model in Example 2: ● potential failure element; ▬▬ rigid body; —— structure element.

Table 2. Known data for Example 2

Failure element number	Cross-section area (m^2)	Inertia moment (m^4)	Mean value of capacity ($kN\,m^{-1}$)	Length of rigid body (m)	Mean value of loads (kN)
01, 02	0.126	0.187	38,950.0	1.0 4.2	$L_1 = 2800$
03, 04	0.114	0.111	25,640.0	2.5 2.5	$L_2 = -2790$
05, 06	0.088	0.042	12,850.0	3.1 1.1	$L_3 = 1910$
07, 08	0.088	0.043	13,540.0	6.0 1.1	$L_4 = 820$
09, 10	0.033	0.013	6730.0	1.8 1.9	$L_5 = -840$
11, 12	0.088	0.042	12,850.0	1.1 2.4	$L_6 = 590$
13, 14	0.078	0.042	13,490.0	1.1 2.4	$L_7 = -101$
15, 16	0.100	0.043	12,910.0	2.3 2.4	$L_8 = -450$
17, 18	0.100	0.044	12,700.0	0.0 2.4	$L_9 = -3570$

for the consequence of the plastic hinges. This is due to the consideration of the correlativity of the axial force and transverse shear force in the yield condition, however we have not taken this into account. Nevertheless, the failure probability of the dominant failure paths identified by different methods closely match with each other.

As a result, a total of 38 critical failure paths have been identified. This number is much smaller than that in Ref. [9] in which 292 were found. Furthermore, as illustrated in Fig. 6, only a few out of the 38 critical paths (denoted by the arc of "......") have the failure probability with the quantity of 10^{-2}, all the rest are much smaller than those ones, which have the quantity of 10^{-5}. This situation also matches the result in Ref. [9]. When combining all the critical paths into a series system to calculate the system's reliability, only those which have large failure probability will add a significant contribution to the system failure probability. Thus the model with 38 critical paths identified in the present paper efficiently follows the nodal with 292 paths in Ref. [9].

In the structural system reliability analysis, most of the computational efforts are made to calculate the failure probability. This includes two aspects. First, the failure probability of each branch should be calculated in the process of identifying the critical failure paths. Secondly, after all the critical failure paths have been obtained, the entire system's failure probability should be calculated. With the decrease of the critical failure path number, the computational efforts on both of these two aspects will reduce tremendously. This situation allows the structural system reliability analysis be practically applied in engineering rather than just in the theoretical stage.

6. CONCLUDING REMARK

From the above discussion, we can see that the critical failure probability which is based on the idea of problem-solving in A.I. can be used to substitute the experiential constant of $\Delta\beta_i$ in the β-unzipping method at each level. It can be used to rationalize the process of identifying the critical failure paths and also provides a efficient method to realize the structural system reliability analysis for practical engineering structures.

The problem of not identifying all the machanisms in the mechanism combination for the traditional β-unzipping method is still unsolved in the present paper. Nevertheless, the dominate failure path which has the largest failure probability is always identified, since in the branching process the failure element with the highest failure probability is always chosen to be branched.

In this paper, all the basic variables are assumed to obey normal distribution. For the basic variables with other distribution types, they can be easily transformed to the normal distribution according to the methods in Refs [1, 4].

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APPENDIX

In this Appendix, some expressions of the revised structural element stiffness matrix and the fictitious node force vectors will be listed.

The local coordinate system of the structural element is shown in Fig. A1. When the bending moment M_z is yielded at the node i, the revised stiffness matrix is

$$
\begin{bmatrix}
\frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{3Ei_z}{l^3} & 0 & 0 & 0 & 0 & 0 & -\frac{3Ei_z}{l^3} & 0 & 0 & 0 & \frac{3Ei_z}{l^3} \\
 & & \frac{12Ei_y}{l^3} & 0 & -\frac{6Ei_y}{q^2} & 0 & 0 & 0 & -\frac{12Ei_y}{l^3} & 0 & -\frac{6Ei_y}{l^2} & 0 \\
 & & & \frac{Gi_x}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{Gi_x}{l} & 0 & 0 \\
 & & & & \frac{4Ei_y}{l} & 0 & 0 & 0 & \frac{6Ei_y}{l^2} & & \frac{2Ei_y}{l} & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 & \\
 & & & & & & & \frac{3Ei_z}{l} & 0 & 0 & 0 & -\frac{3Ei_z}{l^2} \\
 & \text{Sym.} & & & & & & & \frac{12Ei_y}{l^3} & 0 & \frac{6Ei_y}{l^2} & 0 \\
 & & & & & & & & & \frac{Gi_x}{l} & 0 & 0 \\
 & & & & & & & & & & \frac{4Ei_y}{l} & 0 \\
 & & & & & & & & & & & \frac{3Ei_z}{l}
\end{bmatrix}.
$$

The fictitious node force can be written as

$$
\{F\}^e = \left\{ 0 \quad \frac{1.5R_i}{l} \quad 0 \quad 0 \quad 0 \quad R_i \quad 0 \quad -\frac{1.5R_i}{l} \quad 0 \quad 0 \quad 0 \quad 0.5R_i \right\},
$$

where R_i is the capacity of bending moment at node i, i.e. $R_i = M_z$ at node i.

When the bending moment M_z is yielded at the node j, the revised stiffness matrix is

$$
\begin{bmatrix}
\frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{3Ei_z}{l^3} & 0 & 0 & 0 & -\frac{3Ei_z}{l^3} & 0 & \frac{3Ei_z}{l^3} & 0 & 0 & 0 & 0 \\
 & & \frac{12Ei_y}{l^3} & 0 & -\frac{6Ei_y}{l^2} & 0 & 0 & 0 & -\frac{12Ei_y}{l^3} & 0 & -\frac{6Ei_y}{l^2} & 0 \\
 & & & \frac{Gi_x}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{Gi_x}{l} & 0 & 0 \\
 & & & & \frac{4Ei_y}{l} & 0 & 0 & 0 & \frac{6Ei_y}{l^2} & 0 & \frac{2Ei_y}{l} & 0 \\
 & & & & & \frac{3Ei_z}{l} & 0 & -\frac{3Ei_z}{l^2} & 0 & 0 & 0 & 0 \\
 & & & & & & \frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{3Ei_z}{l} & 0 & 0 & 0 & 0 \\
 & \text{Sym.} & & & & & & & \frac{12Ei_y}{l^3} & 0 & \frac{6Ei_y}{l^2} & 0 \\
 & & & & & & & & & \frac{Gi_x}{l} & 0 & 0 \\
 & & & & & & & & & & \frac{4Ei_y}{l} & 0 \\
 & & & & & & & & & & & 0
\end{bmatrix}.
$$

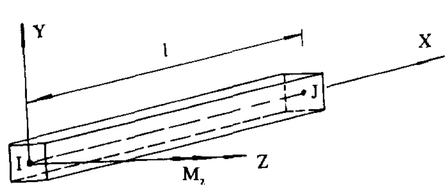

Fig. A1. Local coordinate system of structural element.

The fictitious node force can be written as

$$\{F\}^e = \left\{0 \quad \frac{1.5R_j}{l} \quad 0 \quad 0 \quad 0 \quad 0.5R_j \quad 0 \quad -\frac{1.5R_j}{l} \quad 0 \quad 0 \quad 0 \quad R_j\right\},$$

where R_j is capacity of bending moment at node j, i.e. $R_j = M_z$ at node j. When the bending moment M_z is yielded at both of the node i and node j, the revised stiffness matrix is

$$\begin{bmatrix}
\frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \frac{12Ei_y}{l^3} & 0 & -\frac{6Ei_y}{l^2} & 0 & 0 & 0 & -\frac{12Ei_y}{l^3} & 0 & -\frac{6Ei_y}{l^2} & 0 \\
 & & & \frac{Gi_x}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{Gi_x}{l} & 0 & 0 \\
 & & & & \frac{4Ei_y}{l} & 0 & 0 & 0 & \frac{6Ei_y}{l^2} & 0 & \frac{2Ei_y}{l} & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & \frac{Ea}{l} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 \\
 & \text{Sym.} & & & & & & & \frac{12Ei_y}{l^3} & 0 & \frac{6Ei_y}{l^2} & 0 \\
 & & & & & & & & & \frac{Gi_x}{l} & 0 & 0 \\
 & & & & & & & & & & \frac{4Ei_y}{l} & 0 \\
 & & & & & & & & & & & 0
\end{bmatrix}.$$

The fictitious node force can be written as

$$\{F\}^e = \left\{0 \quad \frac{R_i + R_j}{l} \quad 0 \quad 0 \quad 0 \quad R_i \quad 0 \quad -\frac{R_i + R_j}{l} \quad 0 \quad 0 \quad 0 \quad R_j\right\}.$$